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METRIC DENSITY OF SETS

by

Nathaniel Frizzel Grafton Martin

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Mathematics

Approved:

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1959

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I. INTRODUCTION

A close connection exists between derivatives and indefinite integrals. In establishing this connection for Lebesgue integrals one is led to differentiating set functions, since the Lebesgue integral, or any integral for that matter, is a set function. Different types of derivatives may be defined depending upon the domain of the integrand function. The three most important derivatives are the ordinary derivative, the regular derivative, and the strong derivative. The existence of the strong derivative implies the existence of the regular and ordinary derivatives. If the integrand function of a Lebesgue indefinite integral is defined on Euclidean 1-space these three derivatives are identical. This is not the case however in higher dimensions. A discussion of these derivatives may be found in Munroe (8) and Saks (9). A complete discussion of derivatives of set functions may be found in Hahn and Rosenthal (4).

Lebesgue (7) discovered a very interesting property of measurable sets of points in Euclidean 1-space in the process of differentiating his integral. This result is known as Lebesgue's density theorem.

Essentially the metric density of a measurable set S of points at a point p is the strong derivative at p of the Lebesgue integral of the characteristic function of S. Lebesgue's density theorem states that the strong derivative of this integral is equal to the integrand function almost everywhere, i.e., at every point of the space except possibly for a set of Lebesgue measure zero. Stated this way the result is not too surprising; however, if we probe deeper into the matter we find that the theorem shows that a set of positive Lebesgue measure has the local characteristics of an interval almost everywhere. For, recasting the definition of metric density in terms of measure, we have that the metric density of a measurable set S at any point p is the double limit of the ratio of the measure of the intersection of S with an interval (a, b) containing p to the length of (a, b) as a and b approach p. The density theorem then states that this limit exists at almost every point of S and has the value 1, and this limit exists at almost every point not in S and has the value zero.

Now a question arises concerning the points at which the metric density of S is not zero or one. Is it possible for measurable sets to have density at points and the value of this density be different from zero or one? Goffman (2) has shown that the set of points at which the density of a measurable set exists but is not zero or one must be a set of the first category and constructs a set whose density exists at every point of an F_σ of measure zero and has the value $\frac{1}{2}$. This does not answer the question completely for one might ask "Is it possible for a set to have density of 1/3?" In

Chapter III we show that for any number between 0 and 1 infinitely many sets exist which have density equal to this number at any preassigned point. In Chapter IV Goffman's result is extended by showing that for any F_σ of measure zero and for any number between 0 and 1 there is an infinite number of sets which have density at every point of the F_σ equal to the given number. This is the main result of this paper.

In Chapter III we also consider metric density at a point as a set function by fixing our attention upon one point p and considering the class of all sets which have density at this point. The density at p is then a function on this class of sets which is "almost" a measure function. The results of this portion give some insight into the interrelationships of sets whose density exists at a point. These results are also quite similar to some properties of asymptotic density as obtained by Buck (1) which implies a connection between metric density and asymptotic density.

II. DEFINITIONS, NOTATION, AND PRELIMINARY THEOREMS

Throughout this paper we will use certain well known results and standard definitions. These, together with a few preliminary theorems, are compiled here for convenience.

It will be necessary to speak of sets of points as well as sets whose elements are sets. Sets in general will be denoted by capital Latin letters, e.g., E, F, G, etc. A set whose members are sets will be called a class and classes will be denoted by underlined capital letters, e.g., $\underline{M}$, $\underline{E}$, etc.

If x is an element from a set X then we shall write $x \in X$ and say that x is a member of X or x is in X. If x is not a member of X we shall write $x \notin X$. If A and B are two sets we will say that A is contained in B, denoted by $A \subset B$, if and only if every x which belongs to A is also a member of B. The notation $A \supset B$ will mean $B \subset A$. Two sets are equal if $A \subset B$ and $A \supset B$. A set S is a subset of a set T if S is contained in T.

The notation $\{x: S(x)\}$, where S(x) is a statement concerning x, will mean the set of all elements x for which the statement S(x) is a true statement.

Let A and B be two sets. Then the union, intersection, difference, and symetric difference of A and B are defined respectively by

$$A \cup B = \{x: x \in A \quad \text{or} \quad x \in B\}$$

$$A \cap B = \{x: x \in A \quad \text{and} \quad x \in B\}$$

$$A - B = \{x: x \in A \quad \text{and} \quad x \notin B\}$$

$$A \triangle B = (A \cup B) - (A \cap B).$$

Let X be a set with elements x. For each $x \in X$ let there be associated a set E(x). Then $\bigcup_{x \in X} E(x)$ will mean the set $\{y: y \in E(x) \text{ for some } x \in X\}$, and $\bigcap_{x \in X} E(x)$ will mean the set $\{y: y \in E(x) \text{ for all } x \in X\}$. In particular if X is the set of positive integers, we will write $\bigcup_{x=1}^{\infty} E(x)$ and $\bigcap_{x=1}^{\infty} E(x)$.

If every set throughout a discussion is a subset of a given set X, then the complement of a set S, denoted by S', will mean X-S.

The laws of set algebra will be used freely. These may be found in Halmos (5).

Throughout this paper m will denote Lebesgue linear measure, and most of the sets considered will be subsets of the real line. The real line or Euclidean 1-space will be denoted by R. The null or empty set will be denoted by $\emptyset$.

Let S be a subset of R. The greatest lower bound or infimum of S, denoted by inf S, will mean a real number i which enjoys the following properties:

1. For every $x \in S$, $x \geq i$.
2. For every $\epsilon > 0$ there exists an x^* in S such that $x^* < i + \epsilon$.

The least upper bound or supremum of S, denoted by sup S, will mean a real number s which satisfies the following:

1. For every $x \in S$, $x \leq s$
2. For every $\varepsilon > 0$ there exists an x^* in S such that $x^* + \varepsilon > s$.

The two following theorems will be assumed true.

Theorem 2.1. Every bounded set in R has a least upper bound and a greatest lower bound.

Theorem 2.2. If A and B are subsets of R and A is contained in B then $\sup A \leq \sup B$ and $\inf A \geq \inf B$.

If f is a bounded real valued function defined on a set D, then the set $E = \{f(x) : x \in D\}$ is a bounded subset of R and hence sup E and inf E exist even though D is not necessarily a subset of R. The following theorem, which is well known for real valued functions of a real variable, is proven here in the more general context of a real valued function defined on a set D.

Theorem 2.3. Let f and g be bounded real valued functions defined on D. Then

1. $\inf\{f(x) + g(x) : x \in D\} \geq$
   $\inf\{f(x) : x \in D\} + \inf\{g(x) : x \in D\}$
2. $\inf\{f(x) + g(x) : x \in D\} \leq$
   $\inf\{f(x) : x \in D\} + \sup\{g(x) : x \in D\}$
3. $\sup\{f(x) + g(x) : x \in D\} \geq$
   $\inf\{f(x) : x \in D\} + \sup\{g(x) : x \in D\}$
4. $\sup\{f(x) + g(x) : x \in D\} \leq$
   $\sup\{f(x) : x \in D\} + \sup\{g(x) : x \in D\}$.

Proof.

1. Let $a = \inf\{f(x) + g(x): x \in D\}$, $b = \inf\{f(x): x \in D\}$, and $c = \inf\{g(x): x \in D\}$. For every $x \in D$, $b + c \leq f(x) + g(x)$. Let $\varepsilon > 0$ be given. Then there exists an x^* in D such that $a + \varepsilon > f(x^*) + g(x^*)$. Therefore $a + \varepsilon > b + c$. Since ε was arbitrary $a \geq b + c$.

2. Again let a and b be as in Part 1. Let $c = \sup\{g(x): x \in D\}$. For every $x \in D$, $a \leq f(x) + g(x)$ and $f(x) + g(x) \leq f(x) + c$. Let $\varepsilon > 0$ be given. Then there is an x^* in D such that $b + \varepsilon > f(x^*)$. Therefore $a < b + c + \varepsilon$. Since ε was arbitrary $a \leq b + c$.

3. Let b and c be as in Part 2 and let $a = \sup\{f(x) + g(x): x \in D\}$. Then for every $x \in D$ $a \geq f(x)+ g(x) \geq b + g(x)$. Let $\varepsilon > 0$ be given. Then there is a $x^* \in D$ such that $c < g(x^*) + \varepsilon$. Therefore $a > b + c - \varepsilon$. Since ε was arbitrary, $a \geq b + c$.

4. Let a and c be as in Part 3 and let $b = \sup\{f(x): x \in D\}$. For every $x \in D$, $b + c \geq f(x) + g(x)$. Let $\varepsilon > 0$ be given. Then there is an x^* in D such that $f(x^*)+g(x^*) > a - \varepsilon$. Therefore $b + c > a - \varepsilon$. Since ε was arbitrary $b + c \geq a$.

Let $\{a_n\}$ be a bounded sequence of real numbers. Then $\limsup a_n$ will mean $\inf\{\sup\{a_n: n < m\} : m = 1, 2, \ldots\}$ and $\liminf a_n$ will mean $\sup\{\inf\{a_n: n < m\} : m = 1, 2, \ldots\}$. The following theorem is elementary and will not be proven here.

<u>Theorem 2.4</u>. If $\{a_n\}$ and $\{b_n\}$ are bounded sequences of real numbers then

1. $\liminf a_n \leq \limsup a_n$
2. $\limsup (-a_n) = - \liminf a_n$
3. $\liminf a_n + \liminf b_n \leq \liminf (a_n + b_n)$
4. $\liminf (a_n + b_n) \leq \liminf a_n + \limsup b_n$
5. $\liminf a_n + \limsup b_n \leq \limsup (a_n + b_n)$
6. $\limsup (a_n + b_n) \leq \limsup a_n + \limsup b_n$.

Let $\{I_n\}$ be a sequence of intervals from R and let x be a member of R. Then we will say that $\{I_n\}$ converges to x and write $I_n \to x$ if for every n x is contained in I_n and $\lim m(I_n) = 0$.

Let φ be a bounded real valued function defined on a subclass $\underline{I}$ of the class of all intervals in R. Let $\mathscr{I}(x)$ be the family of all sequences $\{I_n\}$ of intervals from $\underline{I}$ which converge to x. Then $\limsup\limits_{I \to x} \varphi(I)$ and $\liminf\limits_{I \to x} \varphi(I)$ will mean respectively

$$\sup \{ \limsup \varphi(I_n) : \{I_n\} \in \mathscr{I}(x) \}$$

and

$$\inf \{ \liminf \varphi(I_n) : \{I_n\} \in \mathscr{I}(x) \} .$$

Usually we will write

$$\limsup_{I \to x} \varphi(I) = \sup \{ \limsup \varphi(I_n) : I_n \to x \}$$

and

$$\liminf_{I \to x} \varphi(I) = \inf \{ \liminf \varphi(I_n) : I_n \to x \}$$

if there is no confusion as to the domain of the function φ.

In case $\limsup_{I \to x} \varphi(I) = \liminf_{I \to x} \varphi(I)$ we will say that $\lim_{I \to x} \varphi(I)$ exists and has the common value of $\limsup_{I \to x} \varphi(I)$ and $\liminf_{I \to x} \varphi(I)$.

In the next three theorems φ and Θ will represent bounded real valued functions defined on the class $\underline{I}$ of all intervals from R and x will represent a point of R.

<u>Theorem 2.5</u>. $\liminf_{I \to x} \varphi(I) \leqslant \limsup_{I \to x} \varphi(I)$.

<u>Proof</u>. Let $\{I_n\}$ be any sequence of intervals converging to x. Then by Part 1 of Theorem 2.4

$$\liminf \varphi(I_n) \leqslant \limsup \varphi(I_n).$$

Therefore

$$\begin{aligned} &\inf \{ \liminf \varphi(I_n) : I_n \to x \} \\ &\qquad \leqslant \inf \{ \limsup \varphi(I_n) : I_n \to x \} \\ &\qquad \leqslant \sup \{ \limsup \varphi(I_n) : I_n \to x \} . \end{aligned}$$

<u>Theorem 2.6</u>.

1. $\limsup_{I \to x} [-\varphi(I)] = -\liminf_{I \to x} \varphi(I)$
2. $\liminf_{I \to x} \varphi(I) + \liminf_{I \to x} \Theta(I) \leqslant \liminf_{I \to x} [\varphi(I) + \Theta(I)]$
3. $\liminf_{I \to x} [\varphi(I) + \Theta(I)] \leqslant \liminf_{I \to x} \varphi(I) + \limsup_{I \to x} \Theta(I)$
4. $\liminf_{I \to x} [\varphi(I) + \Theta(I)] \leqslant \limsup_{I \to x} \varphi(I) + \liminf_{I \to x} \Theta(I)$
5. $\liminf_{I \to x} \varphi(I) + \limsup_{I \to x} \Theta(I) \leqslant \limsup_{I \to x} [\varphi(I) + \Theta(I)]$

6. $\limsup_{I \to x} \varphi(I) + \liminf_{I \to x} \Theta(I) \leq \limsup_{I \to x} [\varphi(I) + \Theta(I)]$

7. $\limsup_{I \to x} [\varphi(I) + \Theta(I)] \leq \limsup_{I \to x} \varphi(I) + \limsup_{I \to x} \Theta(I).$

<u>Proof</u>.

1. Let $\{I_n\}$ be any sequence of intervals converging to x. Then

$$\limsup [-\varphi(I_n)] = -\liminf \varphi(I_n).$$

Therefore

$$\sup \{\limsup [-\varphi(I_n)] : I_n \to x\}$$
$$= \sup \{-\liminf \varphi(I_n) : I_n \to x\}$$
$$= -\inf \{\liminf \varphi(I_n) : I_n \to x\}.$$

Hence $\limsup_{I \to x} [-\varphi(I)] = -\liminf_{I \to x} \varphi(I).$

2. Let $\{I_n\}$ be any sequence of intervals converging to x. Then

$$\liminf \varphi(I_n) + \liminf \Theta(I_n) \leq \liminf [\varphi(I_n) + \Theta(I_n)].$$

Therefore

$$\inf \{\liminf \varphi(I_n) + \liminf \Theta(I_n) : I_n \to x\}$$
$$\leq \inf \{\liminf [\varphi(I_n) + \Theta(I_n)] : I_n \to x\}.$$

But

$$\inf \{\liminf \varphi(I_n) : I_n \to x\} + \inf \{\liminf \Theta(I_n) : I_n \to x\}$$
$$\leq \inf \{\liminf \varphi(I_n) + \liminf \Theta(I_n) : I_n \to x\}.$$

Hence

$$\liminf_{I \to x} \varphi(I) + \liminf_{I \to x} \Theta(I) \leq \liminf_{I \to x} [\varphi(I) + \Theta(I)].$$

3. For any sequence $\{I_n\}$ converging to x

$$\liminf [\varphi(I_n) + \Theta(I_n)] \leq \liminf \varphi(I_n) + \limsup \Theta(I_n).$$

Therefore

$$\inf\{\liminf[\varphi(I_n) + \Theta(I_n)] : I_n \to x\} \leq \inf\{\liminf \varphi(I_n) + \limsup \Theta(I_n) : I_n \to x\}.$$

However

$$\inf\{\liminf \varphi(I_n) + \limsup \Theta(I_n) : I_n \to x\}$$
$$\leq \inf\{\liminf \varphi(I_n) : I_n \to x\} + \sup\{\limsup \Theta(I_n) : I_n \to x\}.$$

Therefore

$$\liminf_{I \to x} [\varphi(I) + \Theta(I)] \leq \liminf_{I \to x} \varphi(I) + \limsup_{I \to x} \Theta(I).$$

4. This part follows from Part 3 by interchanging the roles of Θ and φ.

5. For any sequence $\{I_n\}$ converging to x

$$\liminf \varphi(I_n) + \limsup \Theta(I_n) \leq \limsup [\varphi(I_n) + \Theta(I_n)].$$

Therefore

$$\sup\{\liminf \varphi(I_n) + \limsup \Theta(I_n) : I_n \to x\}$$
$$\leq \sup\{\limsup [\varphi(I_n) + \Theta(I_n)] : I_n \to x\}.$$

But

$$\sup\{\liminf \varphi(I_n) + \limsup \Theta(I_n) : I_n \to x\}$$
$$\geq \inf\{\liminf \varphi(I_n) : I_n \to x\} + \sup\{\limsup \Theta(I_n) : I_n \to x\}.$$

Therefore

$$\liminf_{I \to x} \varphi(I) + \limsup_{I \to x} \Theta(I) \leq \limsup_{I \to x} [\varphi(I) + \Theta(I)] .$$

6. This part follows from Part 5 by interchanging the roles of φ and Θ.

7. Let $\{I_n\}$ be any sequence of intervals converging to x. Then

$$\limsup [\varphi(I_n) + \Theta(I_n)] \leq \limsup \varphi(I_n) + \limsup \Theta(I_n).$$

Therefore

$$\sup\{\limsup [\varphi(I_n) + \Theta(I_n)] : I_n \to x\}$$
$$\leq \sup\{\limsup \varphi(I_n) + \limsup \Theta(I_n) : I_n \to x\}$$
$$\leq \sup\{\limsup \varphi(I_n) : I_n \to x\} + \sup\{\limsup \Theta(I_n) : I_n \to x\}$$

and hence

$$\limsup_{I \to x} [\varphi(I) + \Theta(I)] \leq \limsup_{I \to x} \varphi(I) + \limsup_{I \to x} \Theta(I).$$

<u>Theorem 2.7</u>. If $\lim_{I \to x} \varphi(I)$ exists then for every sequence $\{I_n\}$ of intervals converging to x, $\lim \varphi(I_n) = \lim_{I \to x} \varphi(I)$. Conversely if for every sequence $\{I_n\}$ of intervals converging to x $\lim \varphi(I_n) = a$ then $\lim_{I \to x} \varphi(I) = a$.

<u>Proof</u>. Suppose first that $\lim_{I \to x} \varphi(I)$ exists. This means that

$$\inf\{\liminf \varphi(I_n) : I_n \to x\} = \sup\{\limsup \varphi(I_n) : I_n \to x\}.$$

But this implies that the sets $\{\limsup \varphi(I_n) : I_n \to x\}$ and

$\{\lim\inf \varphi(I_n)\colon I_n \to x\}$ are identical and contain a single element, say a. Therefore for every sequence $\{I_n\}$ converging to x $\lim\sup \varphi(I_n) = \lim\inf \varphi(I_n) = \lim \varphi(I_n) = a$.

Next assume that $\lim \varphi(I_n) = a$ for every sequence of intervals converging to x. Then for every sequence $\{I_n\}$ converging to x $\lim\sup \varphi(I_n) = \lim\inf \varphi(I_n) = a$ and it follows from this that

$$\{\lim\inf \varphi(I_n)\colon I_n \to x\} = \{\lim\sup \varphi(I_n)\colon I_n \to x\} = \{x\colon x = a\}.$$

Therefore

$$\inf\{\lim\inf \varphi(I_n)\colon I_n \to x\} = \sup\{\lim\sup \varphi(I_n)\colon I_n \to x\}$$
$$= a$$

and consequently

$$\lim_{I \to x} \varphi(I) = a .$$

Let X be a set and let $\underline{R}$ be a class of subsets of X. $\underline{R}$ is a ring if $\underline{R}$ is non null and for every two sets E and F from $\underline{R}$, $E \cup F$ and $E - F$ are also members of $\underline{R}$. If X is a member of $\underline{R}$ then $\underline{R}$ also contains E' whenever $\underline{R}$ contains E. When this is the case $\underline{R}$ is called an algebra. The definitions given here for ring and algebra are due to Halmos (5).

Example 2.1. Let X = R, and let $\underline{S}$ be the class of all subsets of X. Then $\underline{S}$ is an algebra.

Example 2.2. Let X = R, and let I be any bounded interval from R. Let $\underline{S}$ be the class consisting of all subsets of I. Then $\underline{S}$ is a ring but $\underline{S}$ is not an algebra.

Example 2.3. Let $X = R$ and let $\underline{S}$ be the class consisting of all intervals from R. Then $\underline{S}$ is not a ring and consequently not an algebra.

Let $\underline{S}$ be a class of subsets of a set X. A function whose domain of definition is $\underline{S}$ is called a set function. The set functions which we will consider are real valued set functions, i.e., the range of the function is a subset of R.

Let φ be a set function defined on $\underline{S}$. Then φ is said to be finitely additive if and only if for every finite class $\{S_1, S_2, \ldots, S_n\}$ of disjoint sets from $\underline{S}$ whose union is a member of $\underline{S}$ it is true that $\varphi(\bigcup_{i=1}^{n} S_i) = \sum_{i=1}^{n} \varphi(S_i)$. The function φ is said to be monotone if for every S_1 and S_2 from $\underline{S}$, for which $S_1 \subset S_2$, it is true that $\varphi(S_1) \leq \varphi(S_2)$. The function φ is subtractive if for every S_1 and S_2 from $\underline{S}$, for which $S_1 \subset S_2$, $\varphi(S_2 - S_1) = \varphi(S_2) - \varphi(S_1)$. The function φ is countably additive if and only if for every disjoint sequence $\{S_n\}$ of sets from $\underline{S}$ whose union is in $\underline{S}$, $\varphi(\bigcup_{n=1}^{\infty} S_n) = \sum_{n=1}^{\infty} \varphi(S_n)$. The function φ is finitely subadditive if for every finite class of sets from $\underline{S}$ whose union is in $\underline{S}$, $\varphi(\bigcup_{i=1}^{n} S_i) \leqslant \sum_{i=1}^{n} (S_i)$. The function φ is countably subadditive if for every sequence $\{S_n\}$ of sets from $\underline{S}$ whose union is in $\underline{S}$, $\varphi(\bigcup_{n=1}^{\infty} S_n) \leqslant \sum_{n=1} \varphi(S_n)$.

A set function μ defined on a ring $\underline{R}$ is a measure if and only if μ is an extended real valued, countably additive, non-negative set function. A set function μ defined on a ring is a finitely additive measure if it is a non-negative, extended real valued, finitely additive set function defined on a ring. Lebesgue measure is an example of a measure and Jordan Content is an example of a finitely additive measure.

An open interval in R will be denoted by (a, b) and will mean the set $\{x: a < x < b\}$. A closed interval will be denoted by [a, b] and will mean the set $\{x: a \leq x \leq b\}$. An F_σ is a set of points which is the union of a countable number of closed sets.

III. METRIC DENSITY AT A POINT

All sets in this section will be Lebesgue measurable subsets of the real line R.

Definition 3.1. The relative measure of a set S in an interval I, denoted by $\rho(S:I)$, will mean the ratio of the measure of $S \cap I$ to the measure of I.

Some obvious properties of the relative measure of S in I are contained in the following proposition.

Proposition 3.1.

1. For a given measurable set S, $\rho(S:I)$ is defined for all bounded intervals.
2. If $m(S \cap I) = 0$ then $\rho(S:I) = 0$
3. If $S \supset I$ then $\rho(S:I) = 1$
4. For every I $0 \leq \rho(S:I) \leq 1$.
5. For every I $\rho(S':I) = 1 - \rho(S:I)$.

Definition 3.2. Let S be a subset of R and let x be any point of R. The upper metric density of S at x, denoted by $\overline{D}_x(S)$, is defined by

$$\overline{D}_x(S) = \limsup_{I \to x} \rho(S:I).$$

The lower metric density of S at x, denoted by $\underline{D}_x(S)$ is defined by

$$\underline{D}_x(S) = \liminf_{I \to x} \rho(S:I).$$

In case $\underline{D}_x(S) = \overline{D}_x(S)$ the common value is denoted by $D_x(S)$

and is called the metric density of S at x. Thus $D_x(S) = \lim_{I \to x} \rho(S:I)$ provided this limit exists. A necessary and sufficient condition that this limit exist is given by Theorem 2.7.

Example 3.1. Let $S = \{x: 0 < x \leqslant 1\}$. Then for y = 0 or 1 $\overline{D}_y(S) = 1$ and $\underline{D}_y(S) = 0$ and thus the density of S at 0 and 1 does not exist. If y is such that $0 < y < 1$, then $D_y(S) = 1$ and if $y > 1$ or $y < 0$ then $D_y(S) = 0$.

In this example S is a set which contains a point at which the metric density does not exist, and S' contains a point at which the metric density does not exist. Notice also that for every point of R at which the metric density is zero there exists an interval containing this point which contains no points of S. From Part 2 of Proposition 3.1 we see that if there is some interval about x which contains only a subset of S of measure zero then the metric density of S at x is zero. In the next example every interval about 0 contains a subset of S of positive measure, and the metric density of S at 0 is zero.

Example 3.2. Let $I_n = (\frac{1}{n}, \frac{1}{n} + \frac{1}{2^n})$ and define S by $S = \bigcup_{n=1}^{\infty} I_n$.

Example 3.3. Let T = S'. In this example the set T has density of 1 at 0 and every interval about 0 contains a subinterval containing no points of T.

In these examples we have sets whose density exists at a point and has the value 0 or 1 and whose density fails to exist at a point.

<u>Lemma 3.1</u>. Let x_o be any point in R, and let S be any measurable subset of R whose density exists at x_o. Let p be any real number and let S + p denote the set $\{y: y = x + p, x \in S\}$. Then the density of S + p exists at $x_o + p$ and has the value $D_{x_o}(S)$.

<u>Proof</u>. Let $\{I_k\}$ be any sequence of intervals which converges to $x_o + p$. Then the sequence $\{I_k - p\}$, where $I_k - p = \{x: x - p, x \in I_k\}$, is a sequence of intervals converging to x_o. Then,

$$\begin{aligned} \rho[(S + p): I_k] &= \frac{m[(S + p) \cap I_k]}{m(I_k)} \\ &= \frac{m[S \cap (I_k - p)]}{m(I_k - p)} \\ &= \rho[S:(I_k - p)] . \end{aligned}$$

Therefore, since $D_{x_o}(S)$ exists,

$$D_{x_o}(S) = \lim \rho[S:(I_k - p)] = \lim \rho[(S + p): I_k] .$$

Hence, for every sequence $\{I_k\}$ which converges to $x_o + p$, $\lim \rho[(S + p): I_k] = D_{x_o}(S)$. The lemma follows by application of Theorem 2.7.

Theorem 3.1. Let a be any point in R and let δ be any real number with $0 \leq \delta \leq 1$. Then there exists a set G whose density exists at a and has the value δ.

Proof. First suppose $0 < \delta < 1$. For each positive integer n let

$$I_n^R = \left\{x: \frac{1}{n+1} < x < \frac{1}{n}\right\}, \quad I_n^L = \left\{x: -\frac{1}{n} < x < -\frac{1}{n+1}\right\}$$

and denote by G_n^R any open set contained in I_n^R for which $m(G_n^R) = \delta\, m(I_n^R)$. Similarly let G_n^L be any open set contained in I_n^L for which $m(G_n^L) = \delta\, m(I_n^L)$. Define G and K by

$$G = \bigcup_{n=1}^{\infty} (G_n^R \cup G_n^L)$$

$$K = \bigcup_{n=1}^{\infty} [(I_n^R - G_n^R) \cup (I_n^L - G_n^L)].$$

We will show that $D_0(G) = \delta$. Let $J = (h, k)$ be any interval which contains 0, whose length is less than $\frac{1}{2}$. Then, there exists a positive integer N such that

(1) $$\frac{1}{N+1} \leq m(J) \leq \frac{1}{N}$$

and positive integers p and q such that

(2) $$\frac{1}{p} \leq k \leq \frac{1}{p-1}$$

(3) $$-\frac{1}{q-1} \leq h \leq -\frac{1}{q}.$$

Denote the open interval $(\frac{1}{p}, k)$ by J_R and the open interval $(h, -\frac{1}{q})$ by J_L. Then

$$m(J_R) = k - \frac{1}{p} \leqslant \frac{1}{p(p-1)}$$

and

$$m(J_L) = -\frac{1}{q} - h \leqslant \frac{1}{q(q-1)} \quad .$$

From Inequalities 1 and 2, $p \geqslant N$, and from Inequalities 1 and 3, $q \geqslant N$. Thus

$$\frac{1}{p(p-1)} \leqslant \frac{1}{N(N-1)} \qquad \text{and} \qquad \frac{1}{q(q-1)} \leqslant \frac{1}{N(N-1)} \quad .$$

Therefore $m(E) = m(J_L \cup J_R) \leqslant \frac{2}{N(N-1)}$, and applying Inequality 1 we have

(4) $$\frac{m(E)}{m(J)} \leqslant \frac{2(N+1)}{N(N-1)} \quad .$$

Let $H = [(G \cap J) \cup (K \cap J)] - E$. Then H consists of the disjoint intervals G_n^R, G_m^L, $I_n^R - G_n^R$, $I_m^L - G_m^L$ for $n = p, p+1, \ldots$ and $m = q, q+1, \ldots$. Therefore the interval J may be written as

(5) $$J = H \cup E \cup D$$

where D is a denumerable set composed of the point 0 and end points of the disjoint intervals in H. Since H, E, and D are disjoint

(6) $$m(J) = m(H) + m(E) \; .$$

Now,

$$m(H) = \sum_{n=p}^{\infty} m(I_n^R) + \sum_{n=q}^{\infty} m(I_n^L)$$

and

$$m(G \cap H) = \sum_{n=p}^{\infty} m(G_n^R) + \sum_{n=q}^{\infty} m(G_n^L)$$

$$= \delta\left[\sum_{n=p}^{\infty} m(I_n^R) + \sum_{n=q}^{\infty} m(I_n^L)\right]$$

$$= \delta\, m(H) .$$

Therefore

(7) $$\frac{m(G \cap H)}{m(H)} = \delta .$$

From Equation 6 it follows that

(8) $$\frac{m(G \cap H)}{m(J)} \leqslant \frac{m(G \cap H)}{m(H)} = \delta$$

and it follows from Equation 7 that

(9) $$\frac{m(G \cap H)}{m(J)} = \delta \frac{m(H)}{m(J)} .$$

Division of Equation 6 by m(J) and rearranging terms gives

$$\frac{m(H)}{m(J)} = 1 - \frac{m(E)}{m(J)} .$$

Next application of Inequality 4 gives

$$\frac{m(H)}{m(J)} \geqslant 1 - \frac{2(N+1)}{N(N-1)} .$$

Finally application of the last inequality to Equation 9 gives

(10) $$\frac{m(G \cap H)}{m(J)} \geqslant \delta - \frac{2\ (N+1)}{N(N-1)} .$$

Combination of Inequalities 8 and 10 gives

$$(11) \qquad \delta - \frac{2\delta(N+1)}{N(N-1)} \leq \rho(G \cap H: J) \leq \delta .$$

From Equation 5 it follows that

$$m(G \cap J) = m(G \cap H) + m(G \cap E)$$

$$\leq m(G \cap H) + m(E) .$$

Therefore

$$\frac{m(G \cap J)}{m(J)} \leq \delta + \frac{2(N+1)}{N(N-1)} .$$

Also since $G \cap J \supset G \cap H$, $m(G \cap J) \geq m(G \cap H)$. Thus

$$\frac{m(G \cap J)}{m(J)} \geq \delta - \frac{2\delta(N+1)}{N(N-1)} .$$

Combination of the last two inequalities gives

$$(12) \qquad \delta - \frac{2\delta(N+1)}{N(N-1)} \leq \rho(G: I) \leq \delta + \frac{2(N+1)}{N(N-1)} .$$

For any sequence $I_r \to 0$, the sequence $\{N_r\}$ of integers associated with I_r must approach ∞ . Therefore by 12, $D_0(G) = \delta$.

If $\delta = 1$ let $G = \{ x: -\frac{|a|}{2} < x < \frac{|a|}{2} \}$, and if $\delta = 0$ let $G = \{ x: -\frac{3a}{2} < x < -\frac{a}{2} \}$ if a is positive and $\{ x: -\frac{a}{2} < x < -\frac{3a}{2} \}$ if a is negative.

By Lemma 3.1 the set G + a has density δ at a.

Let a be any real number and let $\underline{E}(a)$ be the class of all sets whose metric density exists at a. The density at a then is a set function defined on $\underline{E}(a)$ onto the closed unit interval. This set function will be denoted by D_a and will be called the point density function. The point density function maps $\underline{E}(a)$ onto the closed unit interval in view of Theorem 3.1. The class $\underline{E}(a)$ does not consist of all measurable sets for the set $\{ x: a < x < a+1 \}$ is not a member of $\underline{E}(a)$.

Proposition 3.2. For every set E contained in $\underline{E}(a)$, $0 \leqslant D_a(E) \leqslant 1$.

Theorem 3.2. A necessary and sufficient condition that a set E be a member of $\underline{E}(a)$ is that $\overline{D}_a(E) + \overline{D}_a(E') = 1$.

Proof. Suppose first that E is a member of $\underline{E}(a)$. Let $\{ I_n \}$ be any sequence of intervals which converges to a. Then by Part 5 of Proposition 3.1

$$\rho(S': I_n) = 1 - \rho(S: I_n)$$

for every n. Therefore

$$\lim \rho(S': I_n) = 1 - \lim \rho(S: I_n)$$

and since $\{ I_n \}$ was an arbitrary sequence $D_a(S) + D_a(S') = 1$. Since E is assumed a member of $\underline{E}(a)$, $D_a(S) = \overline{D}_a(S)$ and $D_a(S') = \overline{D}_a(S')$. Therefore $\overline{D}_a(S) + \overline{D}_a(S') = 1$.

Next suppose that $\overline{D}_a(E) + \overline{D}_a(E') = 1$. Since for all I containing a, $\rho(E: I) + \rho(E': I) = 1$, we have that

$$\liminf_{I \to a} \rho(E:I) + \limsup_{I \to a} \rho(E':I) \geqslant 1$$

and therefore

$$\underline{D}_a(E) + \overline{D}_a(E') \geqslant 1 .$$

Hence

$$\begin{aligned} \overline{D}_a(E') &\geqslant 1 - \underline{D}_a(E) \\ &= \overline{D}_a(E) + \overline{D}_a(E') - \underline{D}_a(E) . \end{aligned}$$

Therefore

$$\overline{D}_a(E) - \underline{D}_a(E) \leqslant 0 .$$

Since $\underline{D}_a(E) \leqslant \overline{D}_a(E)$ we have that E is a member of $\underline{E}(a)$.

Corollary 3.2.1. If E is a member of $\underline{E}(a)$ then E' is a member of $\underline{E}(a)$ and

$$D_a(E') = 1 - D_a(E) .$$

Theorem 3.3. Let $E_1, E_2, \ldots, E_n$ be a disjoint class of sets from $\underline{E}(a)$. Then $\bigcup_{k=1}^{n} E_k$ is a member of $\underline{E}(a)$ and

$$D_a(\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} D_a(E_k) .$$

Proof. Let $\{ I_k \}$ be any sequence of intervals which converges to a. Then for every k, since $E_1, \ldots, E_n$ are disjoint,

$$m\left[\bigcup_{i=1}^{n} (E_i \cap I_k)\right] = \sum_{i=1}^{n} m(E_i \cap I_k) .$$

Therefore

$$\rho\left[(\bigcup_{i=1}^{n} E_i): I_k\right] = \sum_{i=1}^{n} \rho(E_i: I_k)$$

for every k. Since $E_i \in \underline{E}(a)$, $\lim_k \rho(E_i: I_k)$ exists for each i and hence

$$\lim_k \rho\left[(\bigcup_{i=1}^{n} E_i): I_k\right] = \sum_{i=1}^{n} D_a(E_i) .$$

Since this last statement is true for all sequences converging to a, the theorem follows.

Theorem 3.4. If E and F are members of $\underline{E}(a)$ and E is a subset of F, then F-E is a member of $\underline{E}(a)$ and $D_a(F - E) = D_a(F) - D_a(E)$.

Proof. Since $E \subset F$ we may write $F = E \cup (F - E)$. For any sequence $\{I_k\}$ of intervals which converges to a

$$m(F \cap I_k) = m(E \cap I_k) + m[(F - E) \cap I_k] \quad .$$

Therefore

$$\rho(F: I_k) = \rho(E: I_k) + \rho[(F - E): I_k] \quad .$$

Since $\rho(E: I_k)$ is finite

$$\rho[(F - E): I_k] = \rho(F: I_k) - \rho(E: I_k) \quad ,$$

and since $D_a(F)$ and $D_a(E)$ exist

$$\lim \rho[(F - E): I_k] = D_a(F) - D_a(E) \quad .$$

Since this is true for any sequence converging to a, we have

$$D_a(F - E) = D_a(F) - D_a(E) \quad .$$

Corollary 3.4.1. The value of D_a at the null set is zero.

Corollary 3.4.2. If E and F are members of $\underline{E}(a)$ and E is a subset of F then $D_a(F) \geqslant D_a(E)$. Therefore D_a is a monotone set function.

Proof. This follows from Theorem 3.4 and Proposition 3.2.

Theorem 3.5. If $E_1, \ldots, E_n$ is a finite disjoint class of sets from $\underline{E}(a)$ each contained in a set E_o from $\underline{E}(a)$ then

$$\sum_{i=1}^{n} D_a(E_i) \leqslant D_a(E_0)$$

Proof. By Theorem 3.3, $D_a(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} D_a(E_i)$. But $\bigcup_{i=1}^{n} E_i$ is a subset of E_0. Therefore by Corollary 3.4.2 the theorem follows.

Theorem 3.6. The set function D_a is neither countably additive nor countably subadditive.

Proof. For each positive integer n let

$$E_n = \left\{x\colon -\frac{1}{n} < x \leqslant -\frac{1}{n+1} \text{ or } \frac{1}{n+1} \leqslant x < \frac{1}{n}\right\} .$$

Denote by E_0 the union of the E_n. Then $E_0 = (-1, 1) - \{0\}$. Now $\{E_n\}$ is a disjoint sequence of sets and for each n, $D_0(E_n) = 0$. Therefore $\sum_{n=1}^{\infty} D_0(E_n) = 0$. But $D_0(E_0) = 1$. By Lemma 3.1, $\sum_{n=1}^{\infty} D_a(E_n + a) = 0$, but $D_a(E_0 + a) =$

$D_a\left[\bigcup_{n=1} (E_n + a)\right] = 1.$

Theorem 3.7. The class $\underline{E}(a)$ is not closed under intersections.

Proof. For each positive integer n, let

$$R_n = \left\{x\colon \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right) < x < \frac{1}{n}\right\}$$

$$L_n = \left\{x\colon -\frac{1}{n} < x < -\frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right)\right\}$$

$$L_n^* = \left\{x\colon -\frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right) < x < -\frac{1}{n+1}\right\}$$

and define E and F by

$$E = \bigcup_{n=1}^{\infty} (R_n \cup L_n)$$

$$F = \bigcup_{n=1}^{\infty} (R_n \cup L_n^*) .$$

Then by Theorem 3.1, $D_0(E) = D_0(F) = \frac{1}{2}$, but $E \cap F = \bigcup_{n=1}^{\infty} R_n$. Now we will show that $\underline{D}_0(\bigcup_{n=1}^{\infty} R_n) = 0$ and $\overline{D}_0(\bigcup_{n=1}^{\infty} R_n) \geq \frac{1}{2}$. So consequently $E \cap F$ is not a member of $\underline{E}(0)$.

For each positive integer k define I_k to be the closed interval $[-\frac{1}{k}, 0]$. Then $\rho(\bigcup_{n=1}^{\infty} R_n : I_k) = 0$ for each k and it follows that $\underline{D}_0(\bigcup_{n=1}^{\infty} R_n) = 0$.

Next let $I_k = [0, \frac{1}{k}]$. Then for each k

$$\rho(\bigcup_{n=1}^{\infty} R_n : I_k) = \frac{\sum_{n=k}^{\infty} \frac{1}{2}(\frac{1}{n} - \frac{1}{n+1})}{\frac{1}{k}} = \frac{1}{2}$$

and it follows that $\overline{D}_0(\bigcup_{n=1}^{\infty} R_n) \geq \frac{1}{2}$.

Application of Lemma 3.1 gives us that

$$D_a(E + a) = D_a(F + a) = \frac{1}{2} \text{ but } (E + a) \cap (F + a) = \bigcup_{n=1}^{\infty} R_n + a$$

and $\bigcup_{n=1}^{\infty} R_n + a$ is not a member of $\underline{E}(a)$.

Corollary 3.7.1. The class $\underline{E}(a)$ is not closed under unions.

Proof. Suppose $\underline{E}(a)$ were closed under unions. Then for E and F arbitrary members of $\underline{E}(a)$, $E - F = E \cup F - F$ is a member of $\underline{E}(a)$. Thus $E \triangle F$ is contained in $\underline{E}(a)$. However $E \cap F = (E \cup F) - (E \triangle F)$ and the right hand side of this equation is a member of $\underline{E}(a)$ and consequently $E \cap F$ is a member of $\underline{E}(a)$. This is a contradiction of Theorem 3.7.

In view of the preceding results we may say that the domain of D_a is not a ring. Hence D_a is not a finitely additive measure. However we will show next that the class $\underline{E}_o(a)$ which consists of all sets whose density at a is zero does form a ring.

Lemma 3.2. If E and F are members of $\underline{E}_o(a)$ then $E \cup F$ is a member of $\underline{E}_o(a)$.

Proof. For any interval I containing a

$$\rho[(E \cup F): I] \leq \rho(E: I) + \rho(F: I) .$$

Therefore

$$\limsup_{I \to a} \rho[(E \cup F): I] \leq \limsup_{I \to a} \rho(E:I) + \limsup_{I \to a} \rho(F:I)$$

$$= D_a(E) + D_a(F) = 0 .$$

Hence $\overline{D}_a(E \cup F) \leq 0$ and since $\overline{D}_a(E \cup F) \geq 0$ we have $\overline{D}_a(E \cup F) = 0$. Therefore $D_a(E \cup F) = 0$.

Theorem 3.8. The class $\underline{E}_o(a)$ is a ring.

Proof. By Theorem 3.4 and Lemma 3.2 $\underline{E}_o(a)$ is closed under arbitrary differences. For if E and F are members of $\underline{E}_o(a)$, $E \cup F$ is a member of $\underline{E}_o(a)$ and F is a subset of $E \cup F$. Thus $E - F = E \cup F - F$ and the right hand side of this equation is a member of $\underline{E}_o(a)$.

Next we shall consider operations between elements of $\underline{E}(a)$ and $\underline{E}_o(a)$.

<u>Theorem 3.9</u>. If E_o is a member of $\underline{E}_o(a)$ and E is a member of $\underline{E}(a)$ then $E_o \cap E$ is a member of $\underline{E}_o(a)$.

<u>Proof</u>. Since $E_o \cap E \subset E_o$ for any interval I which contains a

$$m\left[(E_o \cap E) \cap I\right] \leq m\left[E_o \cap I\right] .$$

Therefore

$$\rho\left[(E_o \cap E) : I\right] \leq \rho(E_o : I) .$$

Thus

$$\limsup_{I \to a} \rho\left[(E_o \cap E) : I\right] \leq \limsup_{I \to a} \rho(E_o : I)$$

and it follows that $\overline{D}_a(E_o \cap E) \leq 0$.

<u>Lemma 3.3</u>. If E_o is a member of $\underline{E}_o(a)$ and E is a member of $\underline{E}(a)$ then $E - E_o$ is a member of $\underline{E}(a)$ and $D_a(E - E_o) = D_a(E)$.

<u>Proof</u>. Let I be any interval containing a. Then

$$m\left[(E - E_o) \cap I\right] \geq m(E \cap I) - m(E_o \cap I)$$

so that

$$\rho\left[(E - E_o) : I\right] \geq \rho(E : I) - \rho(E_o : I) .$$

Therefore

$$\underline{D}_a(E - E_o) = \liminf_{I \to a} \rho\left[(E - E_o) : I\right]$$

$$\liminf_{I \to a} \left[\rho(E : I) - \rho(E_o : I)\right] .$$

But by Theorem 2.6

$$\liminf_{I \to a} [\rho(E:\ I) - \rho(E_o:\ I)] \geqslant \liminf_{I \to a} \rho(E:\ I) + \liminf_{I \to a} [-\rho(E_o: I)]$$

$$= \underline{D}_a(E) - \overline{D}_a(E_o) \ .$$

Since $E_o \in \underline{E}_o(a)$, $\overline{D}_a(E_o) = 0$, and we have

$\underline{D}_a(E - E_o) \geqslant D_a(E)$.

Next write $E_o \cup E = (E - E_o) \cup E_o$. Then for any interval I containing a,

$$\rho[(E_o \cup E):\ I] = \rho[(E - E_o):\ I] + \rho(E_o:\ I)$$

or

$$\rho[(E - E_o):\ I] = \rho[(E_o \cup E):\ I] - \rho(E_o:\ I).$$

Therefore

$$\limsup_{I \to a} \rho[(E - E_o):I] \leqslant \limsup_{I \to a} \rho[(E_o \cup E):I] - \liminf_{I \to a} \rho(E_o:I)$$

$$= \overline{D}_a(E_o \cup E) - \underline{D}_a(E_o) \ .$$

Since $\underline{D}_a(E_o) = 0$ we have $\overline{D}_a(E - E_o) \leqslant \overline{D}_a(E_o \cup E)$.

Now $m\,[(E \cup E_o) \cap I] \leqslant m(E \cap I) + m(E_o \cap I)$ so that

$$\rho[(E \cup E_o):\ I] \leqslant \rho(E:\ I) + \rho(E_o:\ I)$$

for all I containing a. Hence

$$\limsup_{I \to a} \rho[(E \cup E_o):I] \leqslant \limsup_{I \to a} \rho(E:I) + \limsup_{I \to a} \rho(E_o:\ I)$$

and consequently $\overline{D}_a(E \cup E_o) \leqslant \overline{D}_a(E)$. Therefore $\overline{D}_a(E - E_o) \leqslant \overline{D}_a(E \cup E_o) \leqslant D_a(E)$. This statement together with $\underline{D}_a(E - E_o) \geqslant D_a(E)$ gives the required result.

Lemma 3.4. If E_o is a member of $\underline{E}_o(a)$ and E is a member of $\underline{E}(a)$ then $E_o - E$ is in $\underline{E}_o(a)$.

Proof. Since $E_o - E = E_o \cap E'$ we may apply Corollary 3.2.1 and Theorem 3.9 to obtain

$$D_a(E_o - E) = D_a(E_o \cap E') = 0 .$$

Theorem 3.10. If E is a member of $\underline{E}(a)$ and E_o is a member of $\underline{E}_o(a)$ then $E \triangle E_o$ is in $\underline{E}(a)$ and $D_a(E \triangle E_o) = D_a(E)$.

Proof. This theorem follows from Lemmas 3.3 and 3.4 and Theorem 3.3.

Theorem 3.11. If E_o is a member of $\underline{E}_o(a)$ and E is a member of $\underline{E}(a)$ then $E_o \cup E$ is in $\underline{E}(a)$ and $D_a(E_o \cup E) = D_a(E)$.

Proof. Since $E_o \cup E = (E_o \cap E) \cup (E_o \triangle E)$ and each of the sets $E_o \cap E$ and $E_o \triangle E$ are member of $\underline{E}(a)$ and are disjoint the metric density of $E_o \cup E$ exists at a. Furthermore

$$\begin{aligned} D_a(E_o \cup E) &= D_a(E_o \cap E) + D_a(E_o \triangle E) \\ &= D_a(E) . \end{aligned}$$

From the preceding theorems one sees that the class of sets of metric density zero at a act somewhat like the class of sets of zero Lebesgue measure. Another subclass of $\underline{E}(a)$ of interest is the class of sets which have density 1 at a, i.e., the sets of $\underline{E}(a)$ mapped onto 1 by D_a. Denote this class of sets by $\underline{E}_1(a)$.

Theorem 3.12. If E is a member of $\underline{E}(a)$ and E_1 is in $\underline{E}_1(a)$ then $E_1 \cup E$ and $E_1 \cap E$ are members of $\underline{E}(a)$. Furthermore $D_a(E_1 \cup E) = 1$ and $D_a(E_1 \cap E) = D_a(E)$.

Proof. Let I be any interval which contains a. Then since $(E_1 \cup E) \cap I \supset E_1 \cap I$,

$$\rho[(E_1 \cup E): I] \geqslant \rho(E_1: I) .$$

Therefore

$$\liminf_{I \to a} \rho[(E_1 \cup E): I] \geqslant \liminf_{I \to a} \rho(E_1: I)$$

$$= D_a(E_1) = 1 .$$

Thus $\underline{D}_a(E_1 \cup E) \geqslant 1$. But $\underline{D}_a(E_1 \cup E) \leqslant \overline{D}_a(E_1 \cup E) \leqslant 1$ and we have $D_a(E_1 \cup E) = 1$.

To prove the assertion concerning $E_1 \cap E$ we write $E_1 \cap E = (E_1' \cup E')'$. Since E_1 is a member of $\underline{E}_1(a)$ we have E_1' a member of $\underline{E}_o(a)$, and since E is a member of $\underline{E}(a)$ we have E' a member of $\underline{E}(a)$. Therefore by Theorem 3.11 $E_1' \cup E'$ is a member of $\underline{E}(a)$ and again the complement of $E_1' \cup E'$ is in $\underline{E}(a)$. Then $D_a(E_1 \cap E) = D_a[(E_1' \cup E')'] = 1 - D_a(E_1' \cup E') = 1 - D_a(E') = D_a(E)$.

Lemma 3.5. If E is a member of $\underline{E}(a)$ and E_1 is a member of $\underline{E}_1(a)$ then $E_1 - E$ and $E - E_1$ are members of $\underline{E}(a)$. Furthermore $D_a(E_1 - E) = D_a(E')$ and $D_a(E - E_1) = 0$.

Proof. By Theorem 3.12 and the equation $E_1 - E = E_1 \cap E'$ it follows that $E_1 - E$ is in $\underline{E}(a)$. Also

$$D_a(E_1 - E) = D_a(E_1 \cap E') = D_a(E') .$$

To prove the assertion concerning $E - E_1$ again use $E - E_1 = E \cap E_1'$, Theorem 3.12, and the fact that $D_a(E_1') = 0$.

Then by Theorem 3.9

$$D_a(E - E_1') = D_a(E \cap E_1') = 0 .$$

<u>Theorem 3.13</u>. If E is in $\underline{E}(a)$ and E_1 is in $\underline{E}_1(a)$ then $E \triangle E_1$ is in $\underline{E}$ and $D_a(E \triangle E_1) = D_a(E')$.

<u>Proof</u>. Since $E \triangle E_1 = (E - E_1) \cup (E_1 - E)$ and the sets on the right hand side are disjoint

$$D_a(E \triangle E_1) = D_a(E - E_1) + D_a(E_1 - E)$$
$$= D_a(E') .$$

From the preceding theorems it appears that sets from $\underline{E}_1(a)$ act like intervals containing a as an interior point or like the whole space R.

Next we will consider the class $\underline{E}_0(a) \cup \underline{E}_1(a)$. We will show that this class is a ring and hence the restriction of D_a to this class gives a finitely additive measure, although rather trivial since its only values would be 0 or 1.

<u>Theorem 3.14</u>. The class $\underline{E}_0(a) \cup \underline{E}_1(a)$ is a ring.

<u>Proof</u>. Since $\underline{E}_0(a)$ and $\underline{E}_1(a)$ are disjoint we need consider only three cases.

Case 1. Let $E \in \underline{E}_0(a)$ and $F \in \underline{E}_0(a)$. In this situation $E - F$ and $E \cup F$ are in $\underline{E}_0(a)$ by Theorem 3.8.

Case 2. Let $E \in \underline{E}_1(a)$ and $F \in \underline{E}_1(a)$. Then by Theorem 3.12, $E \cup F$ is in $\underline{E}_1(a)$. By Lemma 3.5, $D_a(E - F) = 0$ and $E - F$ is in $\underline{E}_0(a)$.

Case 3. Let $E \in \underline{E}_0(a)$ and $F \in \underline{E}_1(a)$. Then by Theorem

3.12, $E \cup F$ is in $\underline{E}_1(a)$ and by Lemma 3.5, $D_a(E - F) = 0$ and $D_a(F - E) = D_a(E') = 1$.

The following is a compilation of the results obtained in this section concerning the point density function and the domain of the point density function.

The class of all sets whose density exists at a fixed point is closed under finite disjoint unions and proper differences. This class is not closed under unions and intersections. The class of all sets which are mapped onto zero by the point density function is a ring, and the class of all sets which are mapped onto either zero or one by the point density function is a ring.

The point density function is a finitely additive, monotone, non-negative, bounded, and subtractive set function. The point density function is neither countably additive nor countably subadditive.

For any set E from $\underline{E}(a)$ and any set E_o from $\underline{E}_o(a)$ we have the following:

1. $D_a(E \cup E_o) = D_a(E)$
2. $D_a(E \cap E_o) = 0$
3. $D_a(E - E_o) = D_a(E)$
4. $D_a(E_o - E) = 0$
5. $D_a(E_o \triangle E) = D_a(E)$.

For any set E from $\underline{E}(a)$ and any set E_1 from $\underline{E}_1(a)$ we have the following:

1. $D_a(E \cup E_1) = 1$
2. $D_a(E \cap E_1) = D_a(E)$
3. $D_a(E - E_1) = 0$
4. $D_a(E_1 - E) = D_a(E')$
5. $D_a(E \triangle E_1) = D_a(E')$.

The point density function does not satisfy the conditions of a measure function in two respects. First, the domain of D_a is not a ring and second D_a is not countably additive. It would be desirable to define a measure function in some natural fashion so that the new function is either a restriction or an extension of D_a. It is possible to define a finitely additive measure which is a restriction of D_a. This is done by considering the upper metric density of sets at a.

Let $\underline{M}$ denote the class of all Lebesgue measurable subsets of R. Then $\overline{D}_a$ is defined on $\underline{M}$ onto the closed unit interval. We will show that $\overline{D}_a$ is a finitely subadditive outer measure.

<u>Theorem 3.15</u>. The set function $\overline{D}_a$ is monotone, finitely subadditive, and $\overline{D}_a(\emptyset) = 0$. Furthermore for any two sets from $\underline{M}$ which are a positive distance apart the upper density of the union is equal to the sum of the upper densities of the sets.

<u>Proof</u>. It is obvious that $\overline{D}_a(\emptyset) = 0$. Let $E \subset F$ be two sets from $\underline{M}$. Then for any interval I such that a is a member of I, $\rho(E: I) \leq \rho(F: I)$, and hence $\limsup_{I \to a} \rho(E: I) \leq$

$\limsup_{I \to a} \rho(F: I)$. Therefore $\bar{D}_a(E) \leq \bar{D}_a(F)$ and $\bar{D}_a$ is monotone.

Next let $E_1, E_2, \ldots, E_n$ be any finite class of sets from $\underline{M}$. Then for any interval I containing a

$$\rho\left[\bigcup_{i=1}^{n} E_i : I\right] \leq \sum_{i=1}^{n} \rho(E_i: I)$$

and hence

$$\limsup_{I \to a} \rho\left[\bigcup_{i=1}^{n} E_i : I\right] \leq \limsup_{I \to a} \sum_{i=1}^{n} \rho(E_i: I)$$

$$\leq \sum_{i=1}^{n} \limsup_{I \to a} \rho(E_i: I) .$$

Therefore $\bar{D}_a(\bigcup_{i=1}^{n} E_i) \leq \sum_{i=1}^{n} \bar{D}_a(E_i)$ and $\bar{D}_a$ is finitely subadditive.

Finally let E and F be two sets from $\underline{M}$ which are a positive distance δ apart. Then let I be any interval containing a whose length is less than $\frac{\delta}{4}$. Then for at least one of the sets, say E, $\rho(E: I) = 0$. Therefore

$$\rho[(E \cup F): I] = \rho(F: I)$$

and

$$\bar{D}_a(E \cup F) = \bar{D}_a(F) .$$

But $\bar{D}_a(E) = 0$ so

$$\bar{D}_a(E \cup F) = \bar{D}_a(E) + \bar{D}_a(F) .$$

<u>Definition 3.3</u>. A set E in $\underline{M}$ will be said to have point

density measure if and only if for each set A from $\underline{M}$

$$\bar{D}_a(A) = \bar{D}_a(A \cap E) + \bar{D}_a(A \cap E') \ .$$

Example 3.4. Let a be a given point of R. Define E to be the set $\{x: a - 1 < x < a + 1\}$. Then E has point density measure.

Definition 3.4. The class of all sets which have point density measure will be denoted by $\underline{M}^*$. If E is a member of $\underline{M}^*$ then E will be called point density measurable. The restriction of $\bar{D}_a$ to $\underline{M}^*$ will be called the point density measure and will be denoted by Δ_a.

Theorem 3.16. If E and F are members of $\underline{M}^*$ then $E \cup F$ and E - F are also members of $\underline{M}^*$. Consequently $\underline{M}^*$ is a ring.

Proof. Since E and F are members of $\underline{M}^*$, for each A in M we have

$$\bar{D}_a(A) = \bar{D}_a(A \cap E) + \bar{D}_a(A \cap E')$$

$$\bar{D}_a(A \cap E) = \bar{D}_a(A \cap E \cap F) + \bar{D}_a(A \cap E \cap F')$$

$$\bar{D}_a(A \cap E') = \bar{D}_a(A \cap E' \cap F) + \bar{D}_a(A \cap E' \cap F') \ .$$

Therefore

$$(1)\ \bar{D}_a(A) = \bar{D}_a(A \cap E \cap F) + \bar{D}_a(A \cap E \cap F') + \bar{D}_a(A \cap E' \cap F) + \bar{D}_a(A \cap E' \cap F').$$

If we replace A by $A \cap (E \cup F)$ in Equation 1 we have

$$\bar{D}_a(A \cap (E \cup F)) = \bar{D}_a(A \cap E \cap F) + \bar{D}_a(A \cap E \cap F) + \bar{D}_a(A \cap E' \cap F)$$

$$= \bar{D}_a(A) - \bar{D}_a(A \cap (E \cup F)').$$

Rearrangement of the last equation gives

$$\overline{D}_a(A) = \overline{D}_a(A \cap (E \cap F)) + \overline{D}_a(A \cap (E \cap F)') .$$

To show that $E - F$ is a member of $\underline{M}^*$ we replace A by $A \cap (E - F) = A \cap (E \cap F')$ in Equation 1.

Theorem 3.17. If E is in $\underline{M}^*$ then E is in $\underline{E}(a)$ and $\triangle_a(E) = D_a(E)$.

Proof. Since E is in $\underline{M}^*$ and R is Lebesgue measurable we have

$$\begin{aligned} 1 = \overline{D}_a(R) &= \overline{D}_a(R \cap E) + \overline{D}_a(R \cap E') \\ &= \overline{D}_a(E) + \overline{D}_a(E') . \end{aligned}$$

Therefore, by Theorem 3.2, E is in $\underline{E}(a)$. Hence

$$\triangle_a(E) = \overline{D}_a(E) = D_a(E) .$$

Example 3.5. For each positive integer n let

$$E_n = \left(\frac{1}{n+1}, \frac{1}{2}\left[\frac{1}{n} + \frac{1}{n+1}\right]\right) \cup \left(-\frac{1}{2}\left[\frac{1}{n} + \frac{1}{n+1}\right], -\frac{1}{n+1}\right)$$

and define E as $\bigcup_{n=1}^{\infty} E_n$. Then by Theorem 3.1, $D_0(E) = \frac{1}{2}$ and E is a member of $\underline{E}(0)$. We will show that E is not a member of $\underline{M}^*$.

For each positive integer n take the set A_n to be $(-\frac{1}{n}, -\frac{1}{2}(\frac{1}{n} + \frac{1}{n+1})) \cup (\frac{1}{n+1}, \frac{1}{2}(\frac{1}{n} + \frac{1}{n+1}))$. Then define A as $\bigcup_{n=1}^{\infty} A_n$. Then

$$\overline{D}_0(A) = \frac{1}{2} , \overline{D}_0(A \cap E) = \frac{1}{2} , \overline{D}_0(A \cap E') = \frac{1}{2}$$

and

$$\overline{D}_0(A) \leq \overline{D}_0(A \cap E) + \overline{D}_0(A \cap E') .$$

This example shows that $\underline{M}^*$ is properly contained in $\underline{E}(a)$ and strongly suggests that the only values of $\triangle_a$ are zero and one. This in fact is the case. Even more surprising is the fact that $\underline{M}^*$ is exactly the union of $\underline{E}_0(a)$ and $\underline{E}_1(a)$ and $\triangle_a$ is just the restriction of D_a to this union.

Lemma 3.6. Let E be a member of $\underline{E}(a)$ and let J be any interval with a as one end point. Then $\overline{D}_a(E \cap J) = D_a(E)$.

Proof. Let $D_a(E) = \delta$. Then for any interval I containing a, $E \cap I \supset (E \cap J) \cap I$ so that $\rho(E: I) \geq \rho(E \cap J: I)$. Therefore

$$\limsup_{I \to a} \rho(E: I) \geq \limsup_{I \to a} \rho(E \cap J: I)$$

and

$$\sup\left\{\limsup \rho(E \cap J: I_k): I_k \to a\right\} \leq \delta .$$

Thus for any sequence $\{I_k\}$ which converges to a

$$\limsup \rho(E \cap J: I_k) \leq \delta.$$

Next we will show that there exists a sequence $\{I_k^*\}$ which converges to a and for which

$$\limsup \rho(E \cap J: I_k^*) = \delta .$$

Suppose a is the left end point of J. Denote the right endpoint of J by b. For each positive integer k let $I_k^* = [a, a + \frac{1}{k}(b - a))$. Then I_k^* converges to a, and since

$I_k^* \subset J$ for all k we have

$$\rho(E\colon I_k^*) = \rho(E \cap J\colon I_k^*) .$$

Therefore, $\lim \rho(E \cap J\colon I_k^*) = \lim \rho(E\colon I_k^*) = \delta$.

A similar argument will show that if a is the right end point of J then $\limsup \rho(E \quad J\colon I_n^*) = \delta$, where $I_n^* = (a - \frac{1}{n}(a - b), a)$ and b is the left end point of J.

Lemma 3.7. Let E be a member of $\underline{E}(a)$, J be an open interval with right endpoint at a, and K be a closed interval with left endpoint at a. Define the set A by

$$A = (E' \cap J) \cup (E \cap K) .$$

Then $\overline{D}_a(A) = \max \{D_a(E), D_a(E')\}$.

Proof. Suppose $D_a(E') \leqslant D_a(E) = \delta$. By Lemma 3.6 $\overline{D}_a(E' \cap J) = D_a(E') \leqslant \delta$. Since $A \supset E \cap K$, $\overline{D}_a(A) \geqslant \overline{D}_a(E \cap K) = \delta$.

Next we will show that $\overline{D}_a(A) \leqslant \delta$. Let $\varepsilon > 0$ be given. Since

$$\overline{D}_a(A) = \sup \left\{ \limsup \frac{m(A \cap I_k)}{m(I_k)} : I_k \to a \right\}$$

there exists a sequence $\{ I_k^* \}$ converging to a for which

$$\overline{D}_a(A) < \limsup \frac{m(A \cap I_k^*)}{m(I_k^*)} + \frac{\varepsilon}{2} .$$

For each interval I_k^* contained in $J \cup K$ we have

$$\frac{m(A \cap I_k^*)}{m(I_k^*)} - \delta = \frac{m(K \cap I_k^*)}{m(I_k^*)} \left[\frac{m(E \cap K \cap I_k^*)}{m(K \cap I_k^*)} - \delta \right]$$

$$+ \frac{m(J \cap I_k^*)}{m(I_k^*)} \left[\frac{m(E' \cap J \cap I_k^*)}{m(J \cap I_k^*)} - \delta \right] .$$

Since E is assumed to be a member of $\underline{E}(a)$

$$\lim \frac{m(E \cap K \cap I_k^*)}{m(K \cap I_k^*)} = \delta \quad ,$$

and even though a is not a member of $J \cap I_k^*$ for any k

$$\lim \frac{m(E' \cap J \cap I_k^*)}{m(J \cap I_k^*)} = 1 - \delta$$

since we may replace $J \cap I_k^*$ by $(J \cap I_k^*) \cup \{a\}$. Then

$m[(J \cap I_k^*) \cup \{a\}] = m(J \cap I_k^*)$ and $m\{E' \cap [(J \cap I_k^*) \cup \{a\}]\} =$ $m(E' \cap J \cap I_k^*)$.

Therefore there exists an N_1 such that for all $k > N_1$

$$\frac{m(E \cap K \cap I_k^*)}{m(K \cap I_k^*)} < \delta + \frac{\varepsilon}{2}$$

and there exists an N_2 such that for all $k > N_2$

$$\frac{m(E' \cap J \cap I_k^*)}{m(J \cap I_k^*)} < \delta + \frac{\varepsilon}{2} \quad .$$

Thus for all $k > N = \max\{N_1, N_2\}$

$$\frac{m(E \cap K \cap I_k^*)}{m(K \cap I_k^*)} - \delta < \frac{\varepsilon}{2}$$

and

$$\frac{m(E' \cap J \cap I_k^*)}{m(J \cap I_k^*)} - \delta < \frac{\varepsilon}{2} \ .$$

Consider the sequence $\left\{ \frac{m(A \cap I_n^*)}{m(I_n^*)} \right\}$ where $0 < n < k - N$.

Since $\left\{ \frac{m(A \cap I_k^*)}{m(I_k^*)} \right\}$ and $\left\{ \frac{m(A \cap I_n^*)}{m(I_n^*)} \right\}$ differ by at most a finite number of terms

$$\limsup \frac{m(A \cap I_k^*)}{m(I_k^*)} = \limsup \frac{m(A \cap I_n^*)}{m(I_n^*)} \ .$$

Now

$$\frac{m(A \cap I_n^*)}{m(I_n^*)} - \delta < \frac{m(K \cap I_n^*)}{m(I_n^*)} \cdot \frac{\varepsilon}{2} + \frac{m(J \cap I_n^*)}{m(I_n^*)} \cdot \frac{\varepsilon}{2}$$

$$= \frac{\varepsilon}{2} \ .$$

Therefore

$$\frac{m(A \cap I_n^*)}{m(I_n^*)} < \delta + \frac{\varepsilon}{2}$$

and

$$\limsup \frac{m(A \cap I_n^*)}{m(I_n^*)} \leqslant \delta + \frac{\varepsilon}{2} \ .$$

Therefore

$$\overline{D}_a(A) < \limsup \frac{m(A \cap I_n^*)}{m(I_n^*)} + \frac{\varepsilon}{2}$$

$$\leqslant \delta + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \delta + \varepsilon \ .$$

Since ε was arbitrary $\overline{D}_a(A) \leqslant \delta$.

<u>Theorem 3.18</u>. If E is a member of $\underline{M}^*$ then E is in $\underline{E}_o(a)$

$\cup \underline{E}_1(a)$. Thus the only values of $\triangle_a$ are zero and one.

Proof. Since E is in $\underline{M}^*$, E is in $\underline{E}(a)$. Let $D_a(E) = \delta$ and let $J = (a - 1, a)$, $K = [a, a + 1]$. Define A by

$$A = (E' \cap J) \cup (E \cap K) .$$

By Lemma 3.7, $\overline{D}_a(A) = \max \{\delta, 1 - \delta\}$. By Lemma 3.6

$$\overline{D}_a(A \cap E) = \overline{D}_a(E \cap K) = \delta$$

and

$$\overline{D}_a(A \cap E') = \overline{D}_a(E' \cap J) = 1 - \delta .$$

Since E is in $\underline{M}^*$,

$$\overline{D}_a(A) = \overline{D}_a(A \cap E) + \overline{D}_a(A \cap E') .$$

Therefore

$$\begin{aligned} \max \{\delta, 1 - \delta\} &= \delta + 1 - \delta \\ &= 1 \end{aligned}$$

and it follows that $\delta = 0$ or 1.

Theorem 3.19. The class $\underline{M}^*$ is equal to $\underline{E}_0(a) \cup \underline{E}_1(a)$.

Proof. It is only necessary to show that $\underline{E}_0(a) \cup \underline{E}_1(a)$ is contained in $\underline{M}^*$. Let $E \in \underline{E}_0(a) \cup \underline{E}_1(a)$.

Suppose first that $D_a(E) = 0$. Let A be any Lebesgue measurable set. Since $A \cap E \subset E$,

$$\overline{D}_a(A \cap E) \leqslant \overline{D}_a(E) = 0 ,$$

and it follows that $D_a(A \cap E) = 0$. Let I be any interval containing a. Then

$$m [(A - E) \cap I] \geqslant m(A \cap I) - m(E \cap I)$$

so that

$$\rho[(A \cap E'): I] \geqslant \rho(A: I) - \rho(E: I) .$$

Therefore

$$\limsup_{I \to a} \rho[(A \cap E'): I] \geqslant \limsup_{I \to a} \rho(A: I) - \limsup_{I \to a} \rho(E: I)$$

$$= \bar{D}_a(A) .$$

Hence $\bar{D}_a(A \cap E') \geqslant \bar{D}_a(A)$. Since $A \cap E' \subset A$, $\bar{D}_a(A \cap E') \leqslant \bar{D}_a(A)$ and it follows that $\bar{D}_a(A) = \bar{D}_a(A \cap E')$. Therefore for any measurable set A, $\bar{D}_a(A) = \bar{D}_a(A \cap E) + \bar{D}_a(A \cap E')$.

In case E is in $\underline{E}_1(a)$ we may use the above argument with E replaced by E' to obtain the desired result.

IV. METRIC DENSITY ON AN F_σ

In this section we extend the results of Chapter III by considering sets whose density exists at every point of a given set.

Theorem 4.1. Let there be given an F_σ set Z of measure zero and a real number δ such that $0 < \delta < 1$. Then there is a set whose metric density exists at every point of Z and has the value δ.

Proof. Since Z is an F_σ, it may be written as $\bigcup_{k=1}^{\infty} Z_k$ where for each k, $m(Z_k) = 0$ and Z_k is closed. Furthermore, we may assume without loss of generality that Z is contained in (0, 1).

Let $G_1 = (0, 1)$ and for every positive integer k let G_k be an open set which contains $Z - \bigcup_{n=1}^{k-1} Z_n$. Define T_k to be the set $G_k - Z_k$ and require that $G_{k+1} \subset T_k$. It follows from the manner in which the sequences $\{T_k\}$ and $\{G_k\}$ are defined that each is a monotone non-increasing sequence of sets, i.e., $G_1 \supset G_2 \supset \ldots$ and $T_1 \supset T_2 \supset \ldots$.

Since G_k is an open set and Z_k is closed, T_k is an open set and consequently consists of a countable number of disjoint open intervals I_{kj}. Let k and j be momentarily fixed and consider the interval I_{kj}. Since $m(I_{kj}) < 1$, there exists a positive integer N_{kj} such that

$$(1) \qquad \frac{1}{N_{kj}+1} \leqslant \tfrac{1}{2}m(I_{kj}) \leqslant \frac{1}{N_{kj}} .$$

Let a_{kj} and b_{kj} be respectively the left and right end points of I_{kj}. For each $n \geqslant N_{kj} + 1$ let

$$\alpha_n^{kj} = a_{kj} + \frac{1}{n}$$

$$\beta_n^{kj} = b_{kj} - \frac{1}{n}$$

$$A_n^{kj} = (\alpha_{n+1}^{kj}, \alpha_n^{kj})$$

$$B_n^{kj} = (\beta_n^{kj}, \beta_{n+1}^{kj})$$

$$C^{kj} = I_{kj} - \bigcup_{n=N_{kj}}^{\infty} (A_n^{kj} \cup B_n^{kj}) .$$

The sets A_n^{kj}, B_n^{kj}, and C^{kj} are disjoint and

$$m(I_{kj}) = m\left[\bigcup_{n=N_{kj+1}}^{\infty} (A_n^{kj} \cup B_n^{kj} \cup C^{kj})\right] .$$

For each of the sets A_n^{kj}, B_n^{kj}, C_n^{kj} let AE_n^{kj} be any measurable set contained in A_n^{kj} for which $m(AE_n^{kj}) = \delta m(A_n^{kj})$. Similarly let BE_n^{kj} and CE_n^{kj} be measurable sets contained in B_n^{kj} and C^{kj} respectively such that $m(BE_n^{kj}) = \delta m(B_n^{kj})$ and $m(CE_n^{kj}) = \delta m(C^{kj})$. Let

$$AF_n^{kj} = A_n^{kj} - AE_n^{kj}$$

$$BF_n^{kj} = B_n^{kj} - BE_n^{kj}$$

$$CF_n^{kj} = C^{kj} - CF_n^{kj} .$$

Then

$$m(AF_n^{kj}) = (1 - \delta)m(A_n^{kj})$$

$$m(BF_n^{kj}) = (1 - \delta)m(B_n^{kj})$$

$$m(CF_n^{kj}) = (1 - \delta)m(C^{kj}) .$$

Finally let

$$E_k = \bigcup_{j=1}^{\infty} \bigcup_{n=N_{kj}}^{\infty} [AE_n^{kj} \cup BE_n^{kj} \cup CE_n^{kj}]$$

and

$$F_k = \bigcup_{j=1}^{\infty} \bigcup_{n=N_{kj}}^{\infty} [AF_n^{kj} \cup BF_n^{kj} \cup CF_n^{kj}].$$

Then the sets E_k and F_k are disjoint and their union is T_k.

Returning to the sets G_k, restrict the measure of G_k so that

$$(2) \quad \left\{ \begin{array}{ll} \rho(G_k : AE_n^{k-q\ j}) \leq \frac{1}{n^q} & \rho(G_k : AF_n^{k-q\ j}) \leq \frac{1}{n^q} \\ \rho(G_k : BE_n^{k-q\ j}) \leq \frac{1}{n^q} & \rho(G_k : BF_n^{k-q\ j}) \leq \frac{1}{n^q} \\ \rho(G_k : CE_n^{k-q\ j}) \leq \frac{1}{n^q} & \rho(G_k : CF_n^{k-q\ j}) \leq \frac{1}{n^q} \end{array} \right.$$

for $q = 1, 2, \ldots, k-1$; $j = 1, 2, \ldots$; and $n \geq N_{kj} + 1$, which is possible since Z has measure zero.

Define the set S by

$$S = \bigcup_{k=1}^{\infty} (E_k - G_{k+1}) .$$

We will show that for every x in Z, $D_x(S) = \delta$.

Let x be any element in Z. Then there is a smallest h such that $x \in Z_h \subset G_h$. Let I be an open interval containing x and contained in G_h. Since $m(I) < 1$, there exists a positive integer p such that

$$(3) \qquad \frac{1}{p+1} \leqslant m(I) \leqslant \frac{1}{p} \quad .$$

Since $x \in G_h$, $x \notin T_h$. Therefore, by Inequality 3, if an end point of I falls in A_n^{hj}, B_n^{hj}, or C^{hj} its distance from a_{hj} or b_{hj} cannot exceed $\frac{1}{p}$. Hence if an end point of I is in an AE_n^{hj}, BE_n^{hj}, or CE_n^{hj} the measure of this containing set is less than $\frac{\delta}{p(p+1)}$, and if an end point of I is in an AF_n^{hj}, BF_n^{hj}, or CF_n^{hj} the measure of this containing set is less than $\frac{1-\delta}{p(p+1)}$.

The interval I consists of the following:

1. A set H composed of disjoint intervals A_n^{hj}, B_n^{hj}, C^{hj} where $n \geqslant p$.

2. A set J which consists of two half open intervals, possibly empty, at the ends of I each of whose lengths does not exceed $\frac{1}{p(p+1)}$ so that m(J) does not exceed $\frac{2}{p(p+1)}$.

3. A countable set D which consists of end points of the open intervals in H.

4. A set $N = Z \cap I$.

From Inequality 3 and the maximum measure of J we have that

$$\frac{m(J)}{m(I)} \leqslant \frac{2}{p} \quad . \tag{4}$$

It follows from the definition of H, that

$$m(H) = \sum_{n,j} \left[m(A_n^{hj}) + m(B_n^{hj}) + m(C^{hj}) \right] \quad ,$$

and

$$H \cap E_h = \bigcup_{n,j}^{\infty} \left(AE_n^{hj} \cup BE_n^{hj} \cup CE_n^{hj} \right) \; .$$

Since AE_n^{hj}, BE_n^{hj}, and CE_n^{hj} are disjoint we have that

$$\begin{aligned} m(H \cap E_h) &= \sum_{n,j} \left[m(AE_n^{hj}) + m(BE_n^{hj}) + m(CE_n^{hj}) \right] \\ &= \delta \sum_{n,j} \left[m(A_n^{hj}) + m(B_n^{hj}) + m(C^{hj}) \right] \\ &= \delta \, m(H) \; . \end{aligned}$$

Therefore $\frac{m(H \cap E_h)}{m(H)} = \delta$. We may show in a similar manner that $\frac{m(H \cap F_h)}{m(H)} = 1 - \delta$. Since $H \subset I$, $m(H) \leqslant m(I)$ and consequently

$$\frac{m(H \cap E_h)}{m(I)} \leqslant \frac{m(H \cap E_h)}{m(H)} = \delta \tag{5}$$

$$\frac{m(H \cap F_h)}{m(I)} \leqslant \frac{m(H \cap F_h)}{m(H)} = 1 - \delta \; . \tag{6}$$

The interval I may be written as the disjoint union of H, J, D, and N so that

$$m(I) = m(H) + m(J),$$

and

$$\frac{m(H)}{m(I)} = 1 - \frac{m(J)}{m(I)}$$

$$\geqslant 1 - \frac{2}{p} \quad .$$

Then

$$\frac{m(H \cap E_h)}{m(I)} = \frac{m(H \cap E_h)}{m(H)} \cdot \frac{m(H)}{m(I)} \tag{7}$$

$$\geqslant \delta(1 - \frac{2}{p}) \geqslant \delta - \frac{2}{p}$$

and

$$\frac{m(H \cap F_h)}{m(I)} \geqslant (1 - \delta) - \frac{2}{p} \quad . \tag{8}$$

Thus we have from Inequalities 5, 6, 7, and 8 that

$$\begin{aligned} \delta - \frac{2}{p} \leqslant \frac{m(H \cap E_h)}{m(I)} &\leqslant \delta \\ (1 - \delta) - \frac{2}{p} \leqslant \frac{m(H \cap F_h)}{m(I)} &\leqslant 1 - \delta \quad . \end{aligned} \tag{9}$$

It follows from Inequalities 2 that

$$\begin{aligned} m(G_{k+q} \cap AE_n^{kj}) &\leqslant \frac{1}{n^q}\, m(AE_n^{kj}) \\ m(G_{k+q} \cap BE_n^{kj}) &\leqslant \frac{1}{n^q}\, m(BE_n^{kj}) \\ m(G_{k+q} \cap CE_n^{kj}) &\leqslant \frac{1}{n^q}\, m(CE_n^{kj}) \\ m(G_{k+q} \cap AF_n^{kj}) &\leqslant \frac{1}{n^q}\, m(AF_n^{kj}) \\ m(G_{k+q} \cap BF_n^{kj}) &\leqslant \frac{1}{n^q}\, m(BF_n^{kj}) \\ m(G_{k+q} \cap CF_n^{kj}) &\leqslant \frac{1}{n^q}\, m(CF_n^{kj}) \end{aligned} \tag{10}$$

for $k, j, q = 1, 2, \ldots$ and $n \geqslant N_{kj} + 1$. Now

$$m\left[G_{h+q} \cap (H \cap E_h)\right] = \sum_{n,j} \left[m(AE_n^{hj} \cap G_{h+q}) + m(BE_n^{hj} \cap G_{h+q}) + m(CE_n^{hj} \cap G_{h+q})\right],$$

so applying Inequalities 10 we have

$$m\left[G_{h+q} \cap (H \cap E_h)\right] \leq \sum_{n,j} \frac{1}{n^q}\left[m(AE_n^{hj}) + m(BE_n^{hj}) + m(CE_n^{hj})\right] .$$

But $n \geq p$ so that

(11) $$m\left[G_{h+q} \cap (H \cap E_h)\right] \leq \frac{1}{p^q}\, m(H \cap E_h)$$

for $q = 1, 2, \ldots$. Since $G_{h+q} \supset E_{h+q}$ we have

(12) $$m\left[E_{h+q} \cap (H \cap E_h)\right] \leq \frac{1}{p^q}\, m(H \cap E_h)$$

for $q = 1, 2, \ldots$.

In a similar fashion we can show that

(13) $$m\left[G_{h+q} \cap (H \cap F_h)\right] \leq \frac{1}{p^q}\, m(H \cap F_h)$$

and

(14) $$m\left[E_{h+q} \cap (H \cap F_h)\right] \leq \frac{1}{p^q}\, m(H \cap F_h)$$

for $q = 1, 2, \ldots$.

Now $I \subset G_h$ and $G_{h+q} \subset G_h$ for $q = 1, 2, \ldots$ so

$$I \cap S \supset I \cap \left[E_h - (G_{h+1} \cup G_{h+2} \cup \ldots)\right] .$$

Also

$$I \cap S = \bigcup_{k=1}^{\infty} \left[I \cap E_k - G_{k+1}\right]$$

and since $I \subset G_h \subset G_{h-s}$ for $s = 1, 2, \ldots, h-1$ we have

$$I \cap E_{h-s} - G_{h-s+1} = \emptyset \qquad s = 1, 2, \ldots, h-1 .$$

Therefore

$$I \cap S = \bigcup_{k=h}^{\infty} (I \cap E_k - G_{k+1}) \subset I \cap \bigcup_{k=h}^{\infty} E_k \ .$$

Thus we may write

$$(15) \qquad I \cap [E_h \cap \bigcup_{k=h+1}^{\infty} G_k] \subset I \cap S \subset I \cap \bigcup_{k=h}^{\infty} E_k \ .$$

From Inequalities 12, 14, and 9 we have

$$m\,[E_{h+q} \cap (H \cap E_h)] \leqslant \frac{\delta}{p^q}\, m(I)$$

and

$$m\,[E_{h+q} \cap (H \cap F_h)] \quad \frac{1-\delta}{p^q}\, m(I)$$

for $q = 1, 2, \ldots$. From the manner in which I has been decomposed we may write

$$I = (H \cap E_h) \cup (H \cap F_h) \cup D \cup N \cup J$$

and

$$(16) \quad I \cap \bigcup_{k=h}^{\infty} E_k = (H \cap E_h) \cup \bigcup_{k=h+1}^{\infty} (H \cap E_h \cap E_k) \cup \bigcup_{k=h+1}^{\infty} (H \cap F_h \cap E_k)$$

$$\cup \bigcup_{k=h}^{\infty} [(D \cup N) \cap E_k] \cup \bigcup_{k=h}^{\infty} (E_k \cap J) \ .$$

Therefore

$$m(I \cap \bigcup_{k=h}^{\infty} E_k) \leqslant m(H \cap E_h) + \sum_{k=h+1}^{\infty} m(H \cap E_k \cap E_h)$$

$$+ \sum_{k=h+1}^{\infty} m\,[F_k \cap H \cap E_h] + m(J) \ ,$$

and application of Inequalities 12, 14, and 9 after division by m(I) gives

$$(17) \quad \frac{m(I \cap \bigcup_{k=h}^{\infty} E_k)}{m(I)} \leq \delta + \delta \sum_{q=1}^{\infty} \frac{1}{p^q} + (1 - \delta) \sum_{q=1}^{\infty} \frac{1}{p^q} + \frac{2}{p}$$

$$= \delta + \frac{2}{p} + \sum_{q=1}^{\infty} \frac{1}{p^q}$$

$$< \delta + \frac{3}{p - 1} \quad .$$

Since $\{G_k\}$ is a monotone non-increasing sequence $I \cap [E_h - \bigcup_{k=h+1}^{\infty} G_k] = I \cap (E_h - G_{h+1})$. Furthermore, since $I \supset (H \cap E_h) \cup (H \cap F_h)$, we have that

$$I \cap (E_h - G_{h+1}) \supset [(H \cap E_h) \cup (H \cap F_h)] \cap (E_h - G_{h+1})$$

$$= [(H \cap E_h) \cap (E_h - G_{h+1})] \cup [(H \cap F_h) \cap (E_h - G_{h+1})] \, .$$

But E_h and F_h are disjoint so the second member of the union on the right is null. Therefore

$$I \cap (E_h - \bigcup_{k=h+1}^{\infty} G_k) \supset (H \cap E_h) - (G_{h+1} \cap H \cap E_h)$$

and

$$m[I \cap (E_h - \bigcup_{k=h+1}^{\infty} G_k)] \geq m(H \cap E_h) - m[G_{h+1} \cap H \cap E_h] \quad .$$

Thus if we apply Inequalities 9 and 11 to this last inequality we get

$$(18) \quad m[I \cap (E_h - \bigcup_{k=h+1}^{\infty} G_k)] \geq (\delta - \frac{2}{p}) m(I) - \frac{1}{p}(\delta - \frac{2}{p}) m(I)$$

$$= (\delta - \frac{2}{p} - \frac{\delta}{p} + \frac{2}{p^2}) \, m(I)$$

$$\geq (\delta - \frac{2 + \delta}{p}) \, m(I) \, .$$

In view of Expression 15 and Inequalities 17 and 18 we have

$$\delta - \frac{2 + \delta}{p} \leqslant (S\colon I) \leqslant \delta + \frac{3}{p - 1} ,$$

and as $I \to x$, $p \to \infty$ so that $\lim\limits_{I \to x} \rho(S\colon I) = \delta$.

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VI. ACKNOWLEDGMENT

The author wishes to express his sincere appreciation to Professor Henry P. Thielman for his encouragement and inspiration throughout the author's graduate studies and the preparation of this thesis.